Mathematical induction

Here is a little anecdote about the German mathematician Gauss who, as a pupil at age 8, did not pay attention in class (can you imagine?), with the result that his teacher made him sum up all natural numbers from 1 to 100. The story has it that Gauss came up with the correct answer 5050 within seconds, which infuriated his teacher. How did Gauss do it? Well, possibly he knew that 1+2

$$1 + 2 + 3 + 4 + \cdots + n = \frac{n \cdot (n+1)}{2}$$

for all natural numbers n. 9 Thus, taking n = 100, Gauss could easily calculate:

$$1 + 2 + 3 + 4 + \cdots + 100 = \frac{100 \cdot 101}{2} = 5050.$$

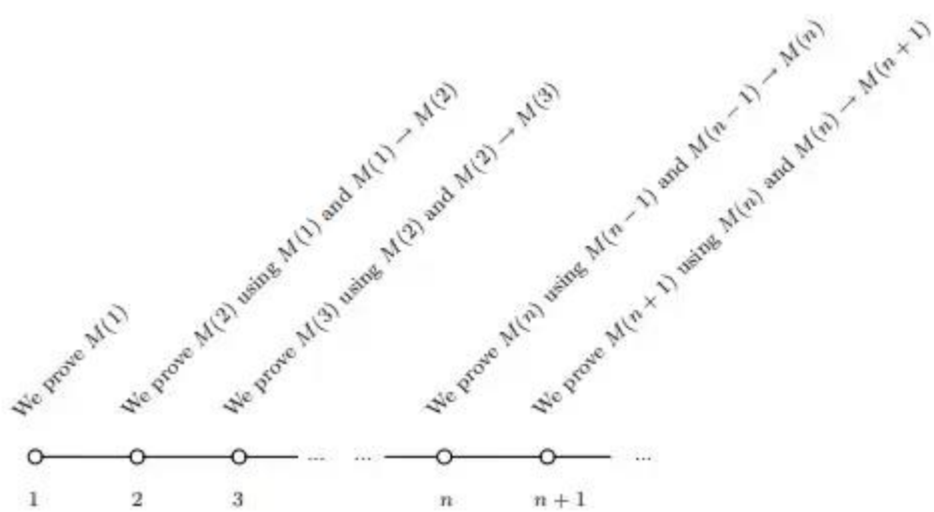
Mathematical induction allows us to prove equations, such as the one in (1.5), for arbitrary n. More generally, it allows us to show that every natural number satisfies a certain property. Suppose we have a property M which we think is true of all natural numbers. We write M(5) to say that the property is true of 5, etc. Suppose that we know the following two things about the property M:

1. Base case: The natural number 1 has property M, i.e. we have a proof of M(1).

2. Inductive step: If n is a natural number which we assume to have property M(n), then we can show that n + 1 has property M(n + 1); i.e. we have a proof of M(n) → M(n + 1).

Theorem

The sum 1+2+3+4+ ⋯ + n equals n · (n + 1)/2 for all natural numbers n.



Proof: We use mathematical induction. In order to reveal the fine structure of our proof we write LHSn for the expression $1 + 2 + 3+4+ \cdots + n$ and RHSn for $n \cdot (n + 1)/2$. Thus, we need to show LHSn = RHSn for all $n \geq 1$.

Base case: If n equals 1, then LHS1 is just 1 (there is only one summand), which happens to equal RHS1 = $1 \cdot (1 + 1)/2$.

Inductive step: Let us assume that LHSn = RHSn. Recall that this assumption is called the induction hypothesis; it is the driving force of our argument. We need to show LHSn+1 = RHSn+1, i.e. that the longer sum $1 + 2 +3+4+ \cdots + (n + 1)$ equals $(n + 1) \cdot ((n + 1) + 1)/2$. The key observation is that the sum $1 + 2 + 3 + 4 + \cdots + (n + 1)$ is nothing but the sum $(1 + 2+3+4+ \cdots + n)+(n + 1)$ of two summands, where the first one is the sum of our induction hypothesis. The latter says that $1+2+3+4+ \cdots + n$ equals $n \cdot (n + 1)/2$, and we are certainly entitled to substitute equals for equals in our reasoning. Thus, we compute

LHSn+1

$=1+2 +3+4+ \cdots + (n + 1)$

= LHSn + (n + 1) regrouping the sum

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Since we successfully showed the base case and the inductive step, we can use mathematical induction to infer that all natural numbers n have the property stated in the theorem above.

Actually, there are numerous variations of this principle. For example, we can think of a version in which the base case is $n = 0$, which would then cover all natural numbers including 0. Some statements hold only for all natural numbers, say, greater than 3. So you would have to deal with a base case 4, but keep the version of the inductive step (see the exercises for such an example). The use of mathematical induction typically suceeds on properties M(n) that involve inductive definitions (e.g. the definition of kl with $l \geq 0$).